# 3. Sequential Estimation.

In section 2. we give out the MLE for mean of Gaussian R.V.

$$\mu_{ML}^{(N)} = \frac{1}{N}\sum_{n} x_n \quad \text{--- the estimation for } N \text{ samples}$$

$$= \frac{1}{N}x_N + \frac{N-1}{N}\mu_{ML}^{(N-1)} \quad \text{.... a weighted estimation involving } \mu_{ML}^{(N-1)}$$

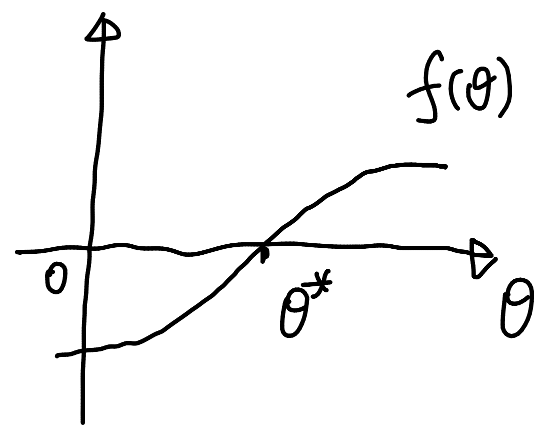
$$= \mu_{ML}^{(N-1)} + \frac{1}{N}(x_N - \mu_{ML}^{(N-1)}) \text{ --- !}$$

It seems that we can have a sequential approaching estimation for $\mu_{ML}^{(\infty)}$. However, this is not always the case, because the estimator may be very complicated. We need some other powerful method.

Robbins - Monro algorithm.

Given R.Vs $z$ & $\theta$, $p(z,\theta)$

def: $f(\theta) = \mathbb{E}(z|\theta) = \int z\, p(z|\theta)\, dz$




$f(\theta)$ is called regression function.

Our goal is to find $\theta^*$ such that $f(\theta^*)=0$.

If we have a large dataset, we can turn to some models to estimate $\theta^*$. But here we will get samples one by one. What we want is to make our estimated $\theta^*$ more and more accurate after each sample.

Assume the conditional variable of $z$ is finite, so

$$\mathbb{E}[(z-\underset{\substack{\uparrow\\ \mathbb{E}(z|\theta)}}{f})^2|\theta]<\infty$$

also. w.l.o.g. we assume $f(\theta)>0$ when $\theta>\theta^*$, $f(\theta)<0$ when $\theta<\theta^*$

The Robbins-Monro procedure defines a sequential successive estimation of $\theta^*$ by

$$\theta^{(N)}=\theta^{(N-1)}+\alpha_{N-1}\, z(\theta^{(N-1)})$$

an observation of $z$ when $\theta=\theta^{(N-1)}$

$\{\alpha_N\}$ is a sequence of positive numbers with

$$\lim_{N\to\infty}\alpha_N=0 \quad \sum\alpha_N=\infty \quad \sum\alpha_N^2<\infty$$ (This is a reasonable step side)

Now let's see how this algorithm could solve MLE. problem.

According to the setting.

$$\frac{\partial}{\partial\theta}\left\{\frac{1}{N}\sum_{n=1}^{N}\ln p(x_n|\theta)\right\}\Bigg|_{\theta_{ML}}=0$$

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\frac{\partial}{\partial\theta}\ln p(x_n|\theta)=\mathbb{E}_x\left[\frac{\partial}{\partial\theta}\ln p(x_n|\theta)\right]$$

frequency $\mapsto$ probability

then use

$$\theta^{(N)}=\theta^{(N-1)}+\alpha_{N-1}\frac{\partial}{\partial\theta^{(N-1)}}\ln p(x_N|\theta^{(N-1)})$$

for Gaussian R.V. $z=\frac{\partial}{\partial\mu_{ML}}\ln p(x|\mu_{ML},\sigma^2)=\frac{1}{\sigma^2}(x-\mu_{ML})$

If we choose $\alpha_N = \frac{\sigma^2}{N}$, we get

$$\theta^{(N)} = \theta^{(N-1)} + \frac{\sigma^2}{N} \cdot \frac{1}{\sigma^2} \cdot (x - \mu_{ML})$$

$$= \theta^{(N-1)} + \frac{1}{N}(x - \mu_{ML})$$ -- same as previous result